Vertex Order Shellings

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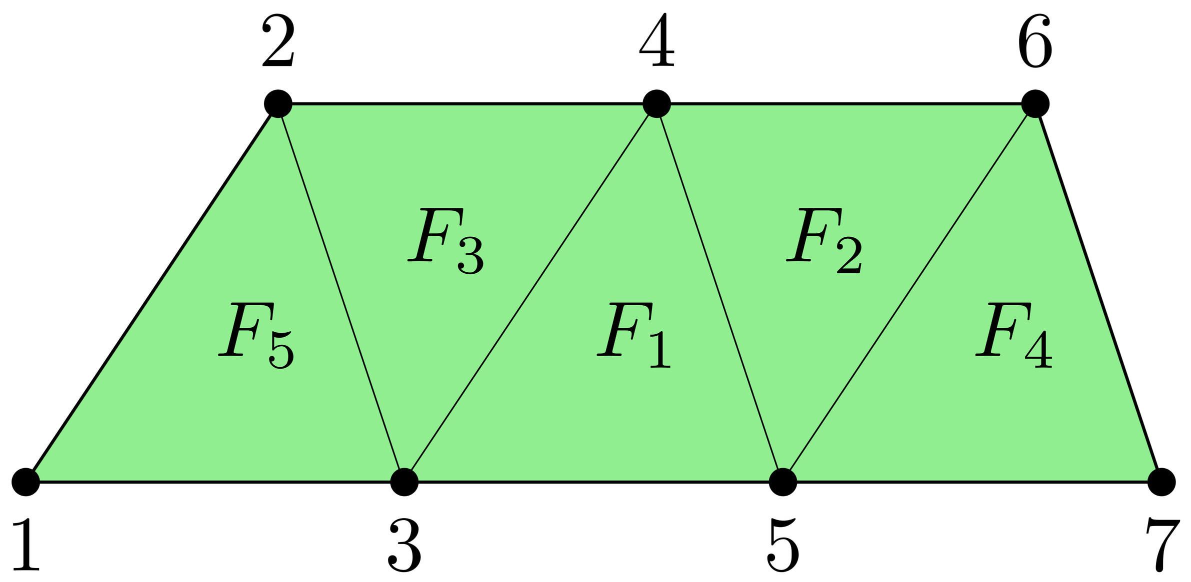
Shellings

Shelling: A total order $F_1, F_2, \ldots$ of the facets of a simplicial complex Δ so that $F_i \setminus \langle F_1, F_2, \ldots, F_{i-1} \rangle$ has a unique minimal face.



Theorem (Björner)

Let Δ be a pure simplicial complex. Then Δ is the indepdence complex of a matroid if and only if, for every ordering $\prec$ of the vertices of Δ, the $\prec$-lexicographic order of the facets is a shelling order.

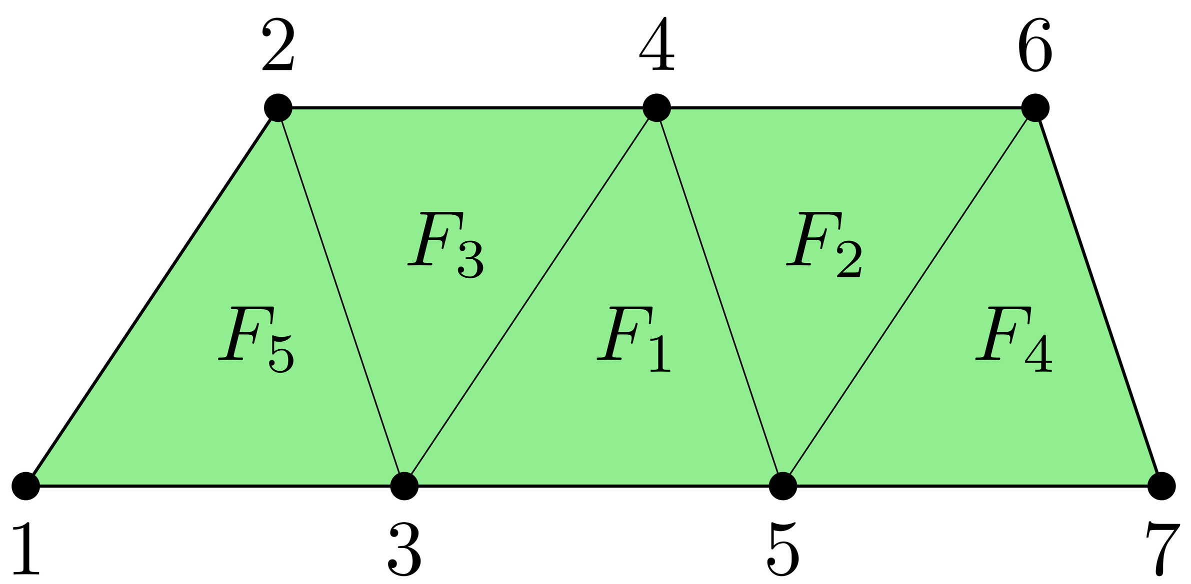
A New Statistic

If $\prec$ makes the $\prec$-lex order a shelling order for Δ, we say $\prec$ is shelling-compatible with Δ.

$$\mathfrak{L}(\Delta) := \frac{1}{n!} \cdot \#\{\text{s.c. orders on } \Delta\}$$

Theorem (Björner, rephrased)

A pure simplicial complex Δ is a matroid independence complex iff $\mathfrak{L}(\Delta) = 1$.

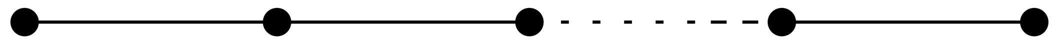


1234567 ✓ 1235674 × 4536271 ✓

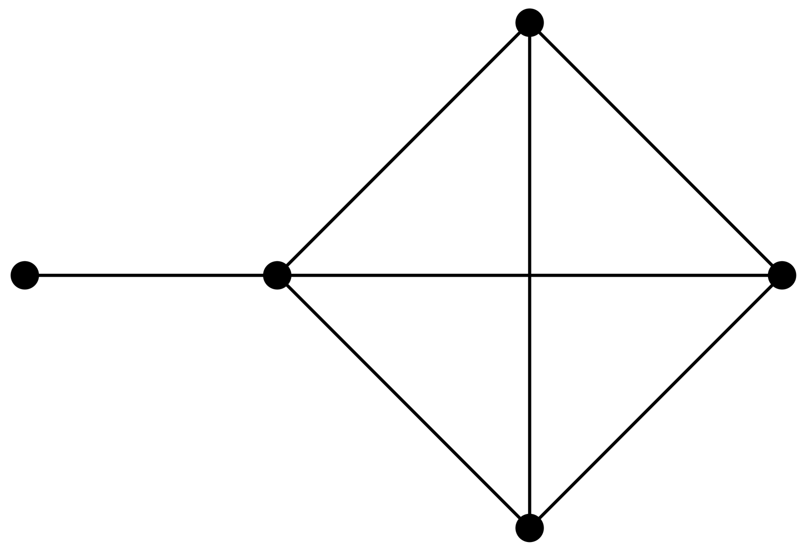
$\mathfrak{L}(\Delta) = \frac{1}{3}$

Interesting Examples

Extremal examples within $0 < \mathfrak{L}(\Delta) < 1$:

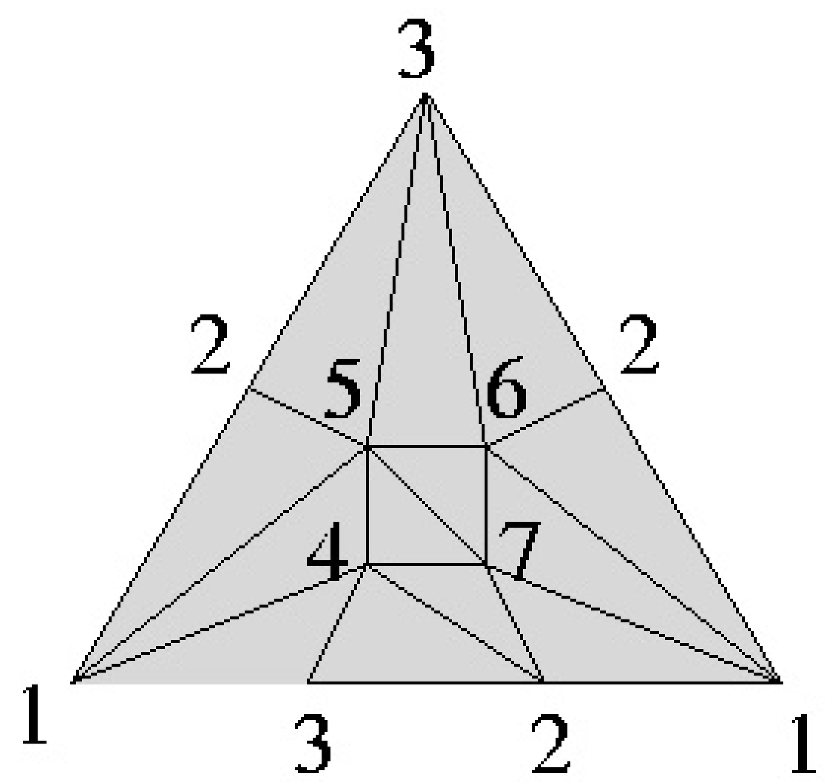


$\lim_{n\to\infty} \mathfrak{L}(P_n) = 0$



$\lim_{n\to\infty} \mathfrak{L}\left(\overline{K_n}\right) = 1$

Relationship to other properties:



Hachimori's complex[a]: not vertex decomposable but $\mathfrak{L} > 0$

The complex with facets

$$\{2,1,4\}, \{3,2,4\}, \{3,2,1\}, \{3,5,4\}, \{3,6,1\}, \{3,6,4\}, \{3,5,1\}, \{5,6,1\}, \{5,2,4\}, \{5,6,2\}, \{12,11,14\}, \{2,12,14\}, \{2,12,11\}, \{2,15,14\}, \{2,1,11\}, \{2,1,14\}, \{2,15,11\}, \{15,1,11\}, \{15,12,14\}, \{15,1,12\}$$

is vertex decomposable but has $\mathfrak{L}(\Delta) = 0$.

[a] infoshako.sk.tsukuba.ac.jp/~hachi/math/library/nonextend_eng.html

More Properties

- $\mathfrak{L}(\Delta_1 * \Delta_2) = \mathfrak{L}(\Delta_1)\mathfrak{L}(\Delta_2)$
- If $\mathfrak{L}(\Delta) > 0$, then the face poset of Δ is EL-shellable.
- For fixed dimension and # vertices,
$$\mathbb{E}(\mathfrak{L}(\Delta)) = \Pr(\text{id is s.c. for uniformly chosen } \Delta).$$
- In particular, when $d = 1$, we have
$$\lim_{n\to\infty} \mathbb{E}(\mathfrak{L}(\Delta)) = 1.$$

Future Directions

Theorem (Coleman–Dochtermann–Geist–Oh)

If $\prec$-revlex is a shelling order for Δ, then:

- *any shelling order of Δ can be completed to a shelling of the d-skeleton of the $(n-1)$-simplex,*
- *Δ is vertex decomposable.*

Question

Can we complete shellings of Δ if $\prec$-lex is a shelling order?

Many other "cryptomorphic" descriptions of matroids exist in literature (independent sets, basis exchange, circuits, rank functions, . . .).

We can define normalized statistics for the degree to which these other properties hold for arbitrary complexes.

Question

What is the relationship between $\mathfrak{L}$ and the corresponding statistics for other matroid axioms?

Question

Let Δ be a pure d-dimensional complex with n vertices. Does there exist a threshold $\Theta(n,d)$ for which

$$\mathfrak{L}(\Delta) > \Theta(n,d) \implies \Delta \text{ vertex decomposable?}$$